Microscopic quantum superpotential in $\mathcal{N} = 1$ gauge theories

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ABSTRACT: We consider the $\mathcal{N} = 1$ super Yang-Mills theory with gauge group U(N), adjoint chiral multiplet X and tree-level superpotential $\operatorname{Tr} W(X)$. We compute the quantum effective superpotential W_{mic} as a function of arbitrary off-shell boundary conditions at infinity for the scalar field X. This effective superpotential has a remarkable property: its critical points are in one-to-one correspondence with the full set of quantum vacua of the theory, providing in particular a unified picture of solutions with different ranks for the low energy gauge group. In this sense, W_{mic} is a good microscopic effective quantum superpotential for the theory. This property is not shared by other quantum effective superpotentials commonly used in the literature, like in the strong coupling approach or the glueball superpotentials. The result of this paper is a first step in extending Nekrasov's microscopic derivation of the Seiberg-Witten solution of $\mathcal{N} = 2$ super Yang-Mills theories to the realm of $\mathcal{N} = 1$ gauge theories.

Contents

1. Introduction and motivations

Since the advent of exact non-perturbative results in four dimensional supersymmetric gauge theories [1], an important line of research is to try to obtain microscopic derivations, from first principles, of the proposed solutions. In the case of $\mathcal{N} = 2$ supersymmetry, one needs to compute instanton contributions for any value of the topological charge and then to sum up the resulting infinite series. Carrying off this project required many years of developments in instanton technology [2–4], culminating in Nekrasov's work [5, 6]. Excellent reviews exist on the subject [7].

A major remaining challenge is to apply Nekrasov's technology to the case of $\mathcal{N} = 1$ gauge theories. Very little work has been done in this subject, with the notable exception of [8]. The main goals are, for example, to obtain a microscopic non-perturbative derivation of the Dijkgraaf-Vafa matrix model approach [9] and of the generalized Konishi anomaly equations [10]. Our aim in the present note is to make the first step in this direction, by explaining in details how and why an instanton analysis can lead to a full microscopic derivation of exact results in $\mathcal{N} = 1$ gauge theories, in spite of the fact that typical vacua are strongly coupled. We are going to derive a microscopic quantum superpotential W_{mic} which has two fundamental properties. First, it can be computed exactly in the instanton approximation, and thus Nekrasov's technology does apply. Second, the solution of the

variational problem $dW_{\mathrm{mic}} = 0$ yields *all* the quantum vacua of the $\mathcal{N} = 1$ theory, including the strongly coupled confining vacua. Of course, at any finite order in the instanton expansion, W_{mic} can only describe the vacua that can be made arbitrarily weakly coupled by adjusting the parameters. The unbroken gauge group in these vacua has only U(1) factors. However, if we use the exact formula for W_{mic}, then we find all the other vacua as well, with non-abelian unbroken gauge groups.

We focus on the well-studied example of the $\mathcal{N} = 1$ theory with U(N) gauge group, an adjoint chiral superfield X and an arbitrary polynomial tree-level superpotential $\mathrm{Tr}\, W(X)$, with

$$W'(z) = \sum_{k=0}^{d} g_k z^k = g_d \prod_{i=1}^{d}(z - w_i)\,. \tag{1.1}$$

In this theory, the classical vacua are labeled as $|N_i\rangle$, with unbroken gauge group U(N_1) $\times \cdots \times$ U(N_d). The integer N_i is equal to the number of eigenvalues of X that are equal to w_i. This is the simplest non-trivial example for the Dijkgraaf-Vafa theory [9], and it displays all the essential features of the problem. It is straightforward to generalize our analysis to other cases.

The plan of the paper is as follows. In section 2, we briefly discuss different types of quantum effective superpotentials, in order to emphasize the special conceptual rôle played by W_{mic}. In section 3, we present the derivation of W_{mic}. In section 4, we study the stationary points of W_{mic} and show that the set of solutions coincide with the full set of quantum vacua of the theory. This provides a full microscopic derivation of the gauge theory expectation values $\langle \mathrm{Tr}\, X^k \rangle$ in any vacuum of the theory, and they coincide with the Dijkgraaf-Vafa prediction. We then conclude and explain future directions of research in section 5.

The contribution of the present paper is mainly to set-up the right conceptual framework to study the $\mathcal{N} = 1$ theories from the microscopic point of view. A very important aspect that we do not address is the calculation of the generalized glueball correlators $\langle \mathrm{Tr}\, W^{\alpha} W_{\alpha} X^k \rangle$, where W^{α} is the chiral vector superfield. These correlators play a central rôle in $\mathcal{N} = 1$ gauge theories and in generalized anomaly equations [10]. Their study from the microscopic point of view is very interesting but technically more involved, and a detailed discussion will appear in forthcoming papers [11, 12].

2. On quantum effective superpotentials

The study of quantum effective superpotentials is an extremely useful point of view in $\mathcal{N} = 1$ gauge theories. There are different types of effective superpotentials one may wish to use, and it is important to understand the technical and conceptual differences between them. We give a brief review of this subject in the present section, in order to put into perspective the properties of the microscopic superpotential W_{mic}.

2.1 On-shell effective superpotential

A central object is the quantum effective superpotential $W_{\mathrm{low}}^{|0\rangle}$, defined by performing the

path integral in a given supersymmetric vacuum $|0\rangle$,

$$e^{i\int \mathrm{d}^4 x\left(2N\,\mathrm{Re}\int\mathrm{d}^2\theta\,W_{\mathrm{low}}^{|0\rangle}(\mathbf{g},q)+D\text{-terms}\right)} = \int_{|0\rangle}\mathrm{d}\mu\,e^{i\mathcal{S}}\,. \tag{2.1}$$

In the above formula, $\mathrm{d}\mu$ denotes the path integral measure (including the ghosts), $\mathcal{S}$ is the super Yang-Mills action, $\mathbf{g}$ denotes collectively the couplings g_k in the tree-level superpotential (1.1), and q is the instanton factor,

$$q = \Lambda^{2N}\,. \tag{2.2}$$

The couplings $\mathbf{g}$ and q have been promoted to arbitrary background chiral superfields. The main property of W_{low} is to yield the on-shell expectation values of the chiral operators by taking the derivative with respect to the coupling constants. If we introduce the operators u_k and glueball superfield S defined by

$$u_k = \mathrm{Tr}\,X^k\,, \quad S = -\frac{1}{16\pi^2 N}\,\mathrm{Tr}\,W^\alpha W_\alpha\,, \tag{2.3}$$

we have

$$\langle 0|u_k|0\rangle = k\frac{\partial W_{\mathrm{low}}^{|0\rangle}}{\partial g_{k-1}}\,, \quad \langle 0|S|0\rangle = q\frac{\partial W_{\mathrm{low}}^{|0\rangle}}{\partial q}\,. \tag{2.4}$$

The quantum superpotential $W_{\mathrm{low}}^{|0\rangle}$ is a fundamentally *on-shell* quantity and it depends strongly on the particular vacuum in which it is computed. To be more precise, W_{low} is generically a multi-valued function of the microscopic couplings $\mathbf{g}$ and q, which means that it can describe several vacua at the same time. For example, if $W(z) = \frac{1}{2}mz^2$, the theory is essentially equivalent to the pure $\mathcal{N} = 1$ gauge theory (after integrating out X). It is well-known that this theory has N vacua, labeled as $|k\rangle$ for $0 \leq k \leq N-1$, and

$$W_{\mathrm{low}}^{|k\rangle} = Nm\,q^{1/N}e^{2i\pi k/N}\,. \tag{2.5}$$

By doing the analytic continuations $q \to qe^{2i\pi}$, we can smoothly interpolate between all the vacua $|k\rangle$ for any k. This is possible because all these vacua are in the same confining phase. It is then more natural to describe the physics in terms of a single multi-valued superpotential $W_{\mathrm{low}}^{|C\rangle} = Nmq^{1/N}$ describing the confining phase $|C\rangle$, instead of using the N possible values (2.5). More generally, when the gauge theory can be realized in several phases, we can associate a multivalued superpotential $W_{\mathrm{low}}^{|\varphi\rangle}$ for each phase $|\varphi\rangle$. The degree of $W_{\mathrm{low}}^{|\varphi\rangle}$ is equal to the number of vacua in the phase $|\varphi\rangle$, and we can interpolate between these vacua by doing analytic continuations. Examples have been studied in [13].

   A particularly interesting feature of the analytic continuations is that, in some examples, they can connect weakly coupled and strongly coupled vacua to each other. This typically happens when fundamental flavors are introduced in the theory. In this case, there is no fundamental distinction between the Higgs and the confining regime (they correspond to the same phase of the theory), and it is possible to interpolate between the Higgs and the confining vacua [14]. In the Higgs regime, the theory is arbitrarily weakly coupled, and thus an instanton calculation is exact. The analytic continuation then allows

to derive exact results in the strongly coupled confining regime, where a direct instanton analysis is not correct (and in particular the small q expansion involves fractional powers of q). This is essentially the philosophy that was used long ago by Shifman and Vainshtein to derive the gluino condensate in pure $\mathcal{N} = 1$ [15], and it is at the basis of a large fraction of our understanding of $\mathcal{N} = 1$ gauge theories.

So instantons can be used in some cases to derive exact results in strongly coupled vacua. We want to know if this idea can be pushed further: is it *always* possible to analyse arbitrary $\mathcal{N} = 1$ vacua starting from an instanton analysis? This is clearly a necessary condition to apply Nekrasov's technology to $\mathcal{N} = 1$ in general and to provide a microscopic derivation of the exact results for this class of theories.

The main drawback of the analysis using W_{low} is that only vacua in the same phase can be connected to each other. The problem clearly comes from the fact that W_{low} is an on-shell quantity. On the other hand, a genuine microscopic quantum superpotential, that can describe all the quantum vacua at the same time, must be an off-shell object. So we need to construct off-shell quantum superpotentials.

2.2 Integrating in

A well-known and very easy way to do that is to "integrate in" some fields starting from W_{low}, which amounts to performing a Legendre transform with respect to the couplings. For example, the glueball superpotential, which plays a prominent rôle in the Dijkgraaf-Vafa approach, is defined as follows. First solve the second equation in (2.4) to express q as a function $q = \hat{q}(S)$ of S. Then define

$$W_{\text{glue}}^{|0\rangle}(s; \mathbf{g}, q) = W_{\text{low}}^{|0\rangle}\big(\mathbf{g}, \hat{q}(s)\big) + \big(\ln q - \ln \hat{q}(s)\big)s\,. \tag{2.6}$$

The superpotential $W_{\text{glue}}(s)$ is an off-shell quantity because the variable s is arbitrary and not necessarily equal to the expectation value of the operator S. By construction, this expectation value in the vacuum $|0\rangle$ can be obtained by solving the "quantum equations of motion"

$$\frac{\partial W_{\text{glue}}^{|0\rangle}}{\partial s}\big(s = \langle 0|S|0\rangle\big) = 0 \tag{2.7}$$

and we have

$$W_{\text{low}}^{|0\rangle} = W_{\text{glue}}^{|0\rangle}\big(s = \langle 0|S|0\rangle\big)\,. \tag{2.8}$$

A priori, W_{glue} depends on a vacuum $|0\rangle$, but it is easy to see that the equation (2.7) actually has several solutions corresponding to different vacua of the same phase. For example, in the case of (2.5), the glueball superpotential is the Veneziano-Yankielowicz superpotential

$$W_{\text{glue}}(S) = S \ln\Big[q\Big(\frac{em}{S}\Big)^N\Big] \tag{2.9}$$

for which (2.7) and (2.8) yields all the vacua $|k\rangle$ for any k. In the case of the theory (1.1), the vacua are labeled by the rank r of the low energy gauge group. For a given rank, the unbroken gauge group is of the form $U(N_1) \times \cdots \times U(N_r)$, and the corresponding classical vacua are of the form $|N_1, \ldots, N_r, 0, \ldots, 0\rangle$. The glueball superpotential can be generalized

in such vacua to a function of r variables s_i corresponding to the glueball fields of each unbroken factor of the gauge group [9, 10]. It is well known that this generalized glueball superpotential describes all the quantum vacua of a given rank. This is interesting because there can be distinct phases of the theory at fixed r. Going off-shell has thus enabled to describe distinct phases with a unique superpotential, albeit for a fixed value of r.

Another possibility is to integrate in the fields u_k defined in (2.3). The resulting superpotential $W_{\rm SC}$ has been used in the literature in the context of the "strong coupling approach" to $\mathcal{N} = 1$, see for example [16]. It has the same qualitative features as the glueball superpotential. It is defined for fixed values of the rank r, in which case r fields (for example $u_1, \ldots, u_r$) are integrated in. This constraint comes from the fact that at rank r, only r of the u_k are independent, and thus the Legendre transform of $W_{\rm low}$ is well-defined only with respect to r (or less) couplings g_k. The quantum equations of motion

$$\frac{\partial W_{\rm SC}}{\partial u_k} = 0, \quad 1 \le k \le r,\tag{2.10}$$

can then be shown to describe all the quantum vacua at fixed r, in a way that is equivalent to the description in terms of the glueball superpotential [16].

So the superpotentials obtained by the integrating in procedure, like $W_{\rm glue}$ or $W_{\rm SC}$, have nice off-shell features (they can describe several phases at the same time), but they are not good enough for our purposes. First, they describe vacua at fixed values of r only, and second it is only in the case $r = N$ (the Coulomb vacuum, which can be made arbitrarily weakly coupled) that they can be computed using an instanton analysis. We are now going to propose a genuine microscopic off-shell superpotential, inspired by Nekrasov's approach, that will not have these drawbacks.

2.3 Microscopic off-shell superpotential

Instead of picking a given vacuum as in (2.1), we consider the euclidean path integral *with arbitrary boundary conditions at infinity* for the chiral adjoint superfield X,

$$X_\infty = \mathrm{diag}(a_1, \ldots, a_N) = \mathrm{diag}\,\mathbf{a}.\tag{2.11}$$

The eigenvalues a_i can be viewed as external chiral superfields on which the path integral depends. We shall use the notation a_i (or $\mathbf{a}$, to denote collectively all the a_is) either for the chiral superfield or for its lowest, scalar, component. The microscopic quantum effective superpotential is then defined by

$$e^{-\int \mathrm{d}^4 x \left(2N\,\mathrm{Re}\int \mathrm{d}^2\theta\, W_{\rm mic}(\mathbf{a};\mathbf{g},q) + D\text{-terms}\right)} = \int_{X_\infty = \mathrm{diag}\,\mathbf{a}} \mathrm{d}\mu\, e^{-\mathcal{S}_{\rm E}}.\tag{2.12}$$

We are using explicitly the euclidean path integral, and $\mathcal{S}_{\rm E}$ is the euclidean super Yang-Mills action. By X_∞, we mean the value of X on the three-sphere at infinity in four dimensional euclidean space.

Several comments on the formula (2.12) are in order. First of all, to be well-defined, we need to introduce an ultraviolet regulator. Since we are going to deal with instantons, it is convenient to use the non-commutative deformation of the theory in order to resolve the
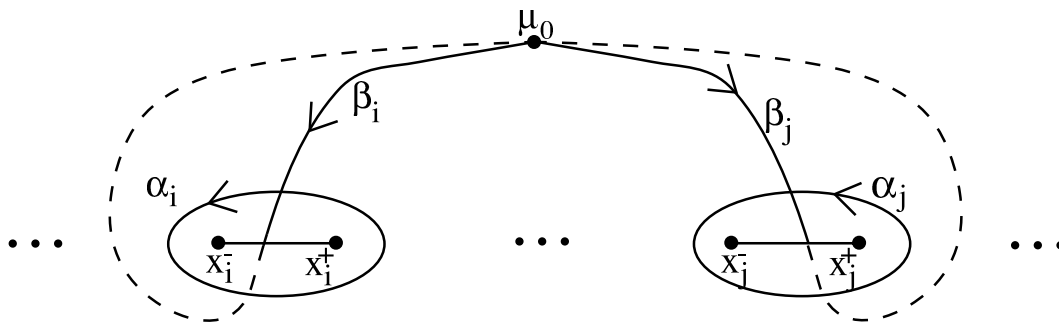
Figure 1: The hyperelliptic curve $\mathcal{C}$ defined by (3.5) with the contours α_i and β_i used in the main text. The open contours β_i go from the point at infinity μ_0 on the first sheet to the point at infinity $\hat{\mu}_0$ on the second sheet.

UV singularities of the instanton moduli space. The chiral observables we are interested in actually do not depend on the non-commutative deformation parameter, which is real, but introducing a non-zero deformation is necessary to obtain well-defined integrals over the moduli space of instantons, with unambiguous definitions of the chiral operators like the u_k in (2.3) for any k. We also need to introduce an infrared regulator, to cut-off the infrared divergence from the integration over space. We use (implicitly) the subtle infrared regulator introduced by Nekrasov [5], which is equivalent to turning on some particular supergravity background (the so-called Ω-background).

The reader might wonder why the path integral (2.12) can depend non-trivially on the boundary conditions **a** when the infrared regulator is removed. Naïvely, one would expect (2.12) to be projected on (2.1), or on a linear combination of contributions corresponding to different vacua. The reason why this does not occur in the supersymmetric theories is that the F-term sector is topological [17], and thus "long distance" can always be pulled to "short distance" by rescaling the metric. The facts that chiral correlators do not depend on the space-time insertion points, and that the integral over the instanton moduli space can be localized on point-like instantons, are other facets of this property.

So we have a natural definition (2.12) for an off-shell microscopic superpotential. Clearly, when $|a_i - a_j| \gg \Lambda$, we can compute W_{mic} in a semiclassical approximation. Since the corresponding instanton series has a finite radius of convergence, the semiclassical approximation is actually exact, and thus $W_{\text{mic}}(\mathbf{a})$ for arbitrary **a** can be obtained from the instanton calculation by analytic continuation.

We now need to understand how to compute W_{mic}, and then to show that the solutions to the equations

$$\frac{\partial W_{\text{mic}}}{\partial a_i} = 0 \tag{2.13}$$

are in one-to-one correspondence with the full set of quantum vacua of the theory (in particular, that these equations describe the vacua for all the possible ranks r).

3. Derivation of W_{mic}

3.1 Off-shell correlators and W_{mic}

From the definition (2.12), it follows that

$$k\frac{\partial W_{\mathrm{mic}}}{\partial g_{k-1}} = \langle \mathbf{a}| \operatorname{Tr} X^k |\mathbf{a}\rangle = u_k(\mathbf{a}, \mathbf{g}, q)\,, \tag{3.1}$$

$$q\frac{\partial W_{\mathrm{mic}}}{\partial q} = \langle \mathbf{a}|S|\mathbf{a}\rangle = S(\mathbf{a}, \mathbf{g}, q)\,, \tag{3.2}$$

where $u_k(\mathbf{a}, \mathbf{g}, q)$ and $S(\mathbf{a}, \mathbf{g}, q)$ are the off-shell expectation values of the operators (2.3) for arbitrary values of the boundary conditions $\mathbf{a}$. These functions can be computed using the results of [5] and [8] as follows.

First, it is shown in [8] (equation (2.17)) that $u_k(\mathbf{a}; \mathbf{g}, q)$ actually does not depend on $\mathbf{g}$,

$$u_k(\mathbf{a}, \mathbf{g}, q) = u_k(\mathbf{a}, q)\,. \tag{3.3}$$

This result is a direct consequence of the localization techniques applied to the integrals over the instanton moduli space. Using (3.3) and (3.1), we deduce that

$$W_{\mathrm{mic}}(\mathbf{a}, \mathbf{g}, q) = \sum_{k=0}^{d} g_k \frac{u_{k+1}(\mathbf{a}, q)}{k+1} + f(\mathbf{a}, q) = \langle \mathbf{a}| \operatorname{Tr} W(X) |\mathbf{a}\rangle + f(\mathbf{a}, q)\,, \tag{3.4}$$

where f is an unknown function of $\mathbf{a}$ and q that does not depend on the couplings $\mathbf{g}$.

When $\mathbf{g} = 0$, the model reduces to the $\mathcal{N} = 2$ gauge theory, and we can use the results of [5]. Let us introduce the Seiberg-Witten curve

$$\mathcal{C}: \; y^2 = P(z)^2 - 4q = \prod_{i=1}^{N}(z - x_i)^2 - 4q = \prod_{i=1}^{N}(z - x_i^-)(z - x_i^+)\,, \tag{3.5}$$

where

$$P_\pm(z) = P(z) \mp 2q^{1/2} = \prod_{i=1}^{N}(z - x_i^\pm)\,. \tag{3.6}$$

The curve (3.5) is hyperelliptic of genus $N - 1$. Various contours and marked points on the curve that we use later in the text are depicted in figure 1. The generating function for the u_k,

$$R(z; \mathbf{a}, q) = \sum_{k \geq 0} \frac{u_k(\mathbf{a}, q)}{z^{k+1}}\,, \tag{3.7}$$

is given by [5, 6]

$$R(z; \mathbf{a}, q) = \frac{P'(z)}{\sqrt{P(z)^2 - 4q}}\,, \tag{3.8}$$

where the parameters x_i entering the curve (3.5) are determined in terms of the a_is by the equations

$$a_i = \frac{1}{2i\pi} \oint_{\alpha_i} z R(z)\, \mathrm{d}z\,. \tag{3.9}$$

Equations (3.7), (3.8) and (3.9) together with (3.4) thus determine W_{mic} up to the function $f(\mathbf{a}, q)$.

Let us note that the a_i defined by (3.9) have been used in many instances in the literature, because they are the natural variables entering into the low energy $\mathcal{N} = 2$ effective action [1]. In particular, they have simple transformation properties under the abelian electric-magnetic duality that plays a central rôle on the $\mathcal{N} = 2$ moduli space. However, presently, we use these variables in a different context. For us, their relevant property is that they precisely coincide with the boundary conditions at infinity for the scalar field X. This is a highly non-trivial result that follows from the explicit all-order instanton calculations of [5, 6].

The q-dependence in f can be determined by using (3.2). A general formula for $S(\mathbf{a}, \mathbf{g}, q)$ has not appeared in the literature, but it can be easily deduced from the analysis of [8]. We do not wish to enter into too much details here, because the analysis of glueball operators $\operatorname{Tr} W^{\alpha} W_{\alpha} X^k$ will be presented elsewhere [11, 12]. However, the case of the operator $S \sim \operatorname{Tr} W^{\alpha} W_{\alpha}$ is particularly simple. The basic formula for S is

$$S(\mathbf{a}, \mathbf{g}, q) = \frac{1}{2\epsilon^2} \Big(\langle \mathbf{a} | \operatorname{Tr} X^2 \operatorname{Tr} W(X) | \mathbf{a} \rangle_{\epsilon} - \langle \mathbf{a} | \operatorname{Tr} X^2 | \mathbf{a} \rangle_{\epsilon} \langle \mathbf{a} | \operatorname{Tr} W(X) | \mathbf{a} \rangle_{\epsilon} \Big), \qquad (3.10)$$

where the limit $\epsilon \to 0$ is understood. The expectation values $\langle \cdots \rangle_{\epsilon}$ are taken for a non-zero Ω-background, the parameter ϵ measuring the strength of this background.[1] The form of the formula (3.10) shows that, to get the glueball operator, the correlators must be computed in the Ω-background including the corrections of order ϵ^2. This is the basic difficulty associated with the glueball operators and also the reason why the analysis of [6], which is limited to the leading order in ϵ, cannot be used straightforwardly. However, in the case of (3.10), there is a huge simplification due to the fact that

$$\frac{1}{2\epsilon^2} \Big(\langle \mathbf{a} | \operatorname{Tr} X^2 \operatorname{Tr} X^k | \mathbf{a} \rangle_{\epsilon} - \langle \mathbf{a} | \operatorname{Tr} X^2 | \mathbf{a} \rangle_{\epsilon} \langle \mathbf{a} | \operatorname{Tr} X^k | \mathbf{a} \rangle_{\epsilon} \Big) = q \frac{\partial \langle \mathbf{a} | \operatorname{Tr} X^k | \mathbf{a} \rangle_{\epsilon}}{\partial q}, \qquad (3.11)$$

for any $k \geq 0$. This equation was derived in [8] and is actually valid for any finite value of ϵ. To give a hint of the origin of (3.11), let us note that the simplifications that allow to derive such an elegant formula are very similar to the ones used in the all-order derivation of the Matone's relations [18] for the $\mathcal{N} = 2$ prepotential [19]. Plugging (3.11) into (3.10), we immediately obtain

$$S(\mathbf{a}, \mathbf{g}, q) = q \frac{\partial \langle \mathbf{a} | \operatorname{Tr} W(X) | \mathbf{a} \rangle_{\epsilon}}{\partial q}. \qquad (3.12)$$

Using (3.4) and (3.2), we see that $f(\mathbf{a}, q) = f(\mathbf{a})$ can depend only on $\mathbf{a}$. We can thus determine f by looking at the classical limit $q \to 0$ for which it is clear that $f = 0$. Note that the classical limit is perfectly smooth since W_{mic} is given by an instanton expansion (this is unlike the classical limit for the glueball superpotential for example; for this reason, W_{glue} can only be determined up to an arbitrary function of the glueball fields s_i by studying the correlators [10]).

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[1] The notation $\hbar$ instead of ϵ is often used in the literature, but we find this rather confusing in particular because ϵ is naturally a complex parameter.

Thus we have derived the fundamental formula

$$W_{\text{mic}}(\mathbf{a}, \mathbf{g}, q) = \langle \mathbf{a} | \operatorname{Tr} W(X) | \mathbf{a} \rangle \,. \tag{3.13}$$

Using (3.7) and (3.8), this is equivalent to

$$W_{\text{mic}}(\mathbf{a}, \mathbf{g}, q) = \frac{1}{2i\pi} \oint_\alpha W(z) R(z; \mathbf{a}, q) \, \mathrm{d}z = \frac{1}{2i\pi} \oint_\alpha \frac{W(z) P'(z)}{\sqrt{P(z)^2 - 4q}} \, \mathrm{d}z \,, \tag{3.14}$$

where the contour $\alpha = \sum_{i=1}^N \alpha_i$.

3.2 Using the U(1)$_{\text{R}}$ symmetry

As emphasized in [20] in the case of the glueball superpotential, R-symmetries put strong constraints on the effective superpotential. We can actually rederive (3.13) by using the U(1)$_{\text{R}}$ symmetry of our model. The charges of the superspace coordinates θ^α, instanton factor q, chiral superfield X, vector superfield W^α, boundary conditions $\mathbf{a}$, couplings $\mathbf{g}$ and superpotential W_{mic} are given in the following table,

$$\begin{array}{c c c c c c c}
 & \theta^\alpha & q & X & W^\alpha & \mathbf{a} & \mathbf{g} & W_{\text{mic}} \\
\text{U}(1)_{\text{R}} & 1 & 0 & 0 & 1 & 0 & 2 & 2 \,.
\end{array} \tag{3.15}$$

Performing an infinitesimal U(1)$_{\text{R}}$ transformation in the path integral (2.12), we obtain

$$2 W_{\text{mic}} = \sum_{k \geq 0} 2 g_k \frac{\partial W_{\text{mic}}}{\partial g_k} \,, \tag{3.16}$$

and using (3.1) we find (3.13) again.

3.3 Relation with the strong coupling approach

The U(1)$_{\text{R}}$ symmetry can be used to constrain the various types of effective superpotentials discussed previously. For example, it yields

$$W_{\text{low}}^{|0\rangle} = \langle 0 | \operatorname{Tr} W(X) | 0 \rangle \,, \tag{3.17}$$

and similarly a very useful constraint is obtained for the Dijkgraaf-Vafa glueball superpotential as explained in [20]. Let us look in more details at the superpotentials W_{SC} used in the strong coupling approach. We denote by $W_{\text{SC}}^{(r)}(u_1, \ldots, u_r)$ the superpotential relevant to the vacua of rank r. The fact that the variables u_k have U(1)$_{\text{R}}$ charge zero makes the $W_{\text{SC}}^{(r)}$ somewhat similar to W_{mic} in the sense that the U(1)$_{\text{R}}$ symmetry also implies that

$$W_{\text{SC}}^{(r)} = \sum_{k \geq 0} g_k \frac{\partial W_{\text{SC}}^{(r)}}{\partial g_k} = \lfloor r | \operatorname{Tr} W(X) | r \rfloor \,. \tag{3.18}$$

By $\lfloor r | \operatorname{Tr} W(X) | r \rfloor$, we mean that the expectation value is computed by taking into account the constraints that correspond to being in a vacuum of rank r. Explicitly, the $u_{k'}$ for $k' > r$ are functions of the u_k for $1 \leq k \leq r$, and thus we have a formula of the form

$$W_{\text{SC}}^{(r)}(u_1, \ldots, u_r) = \sum_{k=1}^r g_k \frac{u_{k+1}}{k+1} + \sum_{k=r+1}^{d+1} g_k \frac{u_{k+1}(u_1, \ldots, u_r; q)}{k+1} \,. \tag{3.19}$$

At the classical level, it is straightforward to write down the constraints that define implicitly the functions $u_k(u_1, \ldots, u_r; q = 0)$. For example, in the simplest $r = 1$ case for which the matrix X is proportional to the identity, we have $u_k = N^{1-k} u_1^k$. The main drawback of the strong coupling approach is that the constraints are not known a priori at the quantum level. They must be postulated based on some physical insights. The correct guess, that originates from [1], is that the rank r vacua are characterized by the factorization condition

$$P(z)^2 - 4q = H_{N-r}(z)^2 R_{2r}(z) \,, \tag{3.20}$$

where H_{N-r} and R_{2r} are polynomials of degrees $N-r$ and $2r$ respectively. This condition is equivalent to the fact that the curve (3.5) degenerates to a genus $r - 1$ surface. Physically, the $\mathcal{N} = 2$ theory then has $N - r$ massless monopoles which can condense when W is turned on, higgsing the low energy gauge group from $\mathrm{U}(1)^N$ to $\mathrm{U}(1)^r$.

Let us assume that $d = N$.[2] Then there exists a rank $r = N$ vacuum, the Coulomb vacuum, corresponding to the unbroken gauge group $\mathrm{U}(1)^N$, described by the superpotential $W_{\mathrm{SC}}^{(N)}$. In the rank N case, the condition (3.20) is trivially satisfied: the variables $u_1, \ldots, u_N$ are independent. It is convenient to use the set of variables $\mathbf{x} = (x_1, \ldots, x_N)$ which, according to (3.8), are related to the u_k for small enough values of k by

$$u_k = \sum_{i=1}^{N} x_i^k \,, \quad 1 \le k \le 2N - 1 \,. \tag{3.21}$$

Equations (3.19) and (3.8) then yield

$$W_{\mathrm{SC}}^{(N)}(\mathbf{x}) = \sum_{i=1}^{N} W(x_i) = \frac{1}{2i\pi} \oint_\alpha \frac{W(z) P'(z)}{\sqrt{P(z)^2 - 4q}} \, \mathrm{d}z \,. \tag{3.22}$$

Comparing with (3.14), we see that

$$W_{\mathrm{mic}}(\mathbf{a}) = W_{\mathrm{SC}}^{(N)}(\mathbf{x}) \,. \tag{3.23}$$

This may look like a rather surprising formula, in view of the important conceptual differences between W_{mic} and $W_{\mathrm{SC}}^{(N)}$. In particular, we have advertised that the solutions to the equations (2.13) are all physical and describe the full set of vacua of the quantum theory. On the other hand, the equations

$$\frac{\partial W_{\mathrm{SC}}^{(N)}}{\partial x_i} = W'(x_i) = 0 \tag{3.24}$$

describe a single vacuum, the weakly coupled Coulomb vacuum. This corresponds to the solution of (3.24) for which all the x_i are distinct and equal to the classical values, $x_i = w_i$ up to permutations. The solutions of (3.24) for which some of the x_i coincide are not physical. This is a trivial artefact of the variables $\mathbf{x}$. To be fully rigorous, we should

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[2]We could assume more generally that $d \ge N$, but this does not bring any new interesting insight.

follow the prescription from the integrating in procedure and use instead the variables $u_1, \dots, u_N$ (3.21). The equations

$$\frac{\partial W_{\text{SC}}^{(N)}}{\partial u_k} = 0 \tag{3.25}$$

then have only one solution corresponding to (3.24) with all the x_i distinct.

The fact that the set of stationary points strongly depends on the variables we use is at the heart of the fundamental difference between W_{mic} and $W_{\text{SC}}^{(N)}$. There is a lot of physics in the choice of the variables, $\mathbf{x}$ or $\mathbf{a}$. This is one of the main point of the present paper. The variables $\mathbf{x}$ enter when one considers the integrating in procedure, as in [16], because of the relation (3.21). On the other hand, a microscopic point of view singles out the variables $\mathbf{a}$, as explained in 3.1.

4. The stationary points of W_{mic}

We are now going to solve the equations (2.13) and prove the claims made earlier in the paper. We use a strategy based on the relationship between W_{SC} and W_{mic}. This has the advantage of exhibiting clearly the differences between the usual integrating in approach and the present microscopic approach. Another derivation of the same results is also possible using generalized Riemann bilinear relations. It will be presented in a forthcoming paper [12].

Let us use (3.23) to rewrite (2.13) as

$$\frac{\partial W_{\text{mic}}}{\partial a_i} = \sum_{i=1}^{N} A_{ij} \frac{\partial W_{\text{SC}}^{(N)}}{\partial x_j} = 0 \,, \tag{4.1}$$

where we have introduced the matrix

$$A_{ij} = \frac{\partial x_j}{\partial a_i} \,. \tag{4.2}$$

The relation between the variables $\mathbf{x}$ and $\mathbf{a}$ is given explicitly by (3.9) and (3.8). The equations (4.1) can be solved in *two* ways:

- Equation (3.24) is satisfied. This case corresponds to the Coulomb vacuum as discussed above.

- Equation (3.24) is *not* satisfied, but $\partial W_{\text{SC}}^{(N)}/\partial x_i$ is an eigenvector of A of zero eigenvalue. This is possible only if the rank of the matrix A is $r < N$. We are going to show that these solutions correspond precisely to the vacua of rank $r < N$.

4.1 Mathematical preliminaries

One-forms h_i. Let us introduce the differential forms on the curve $\mathcal{C}$ (3.5)

$$h_i = \psi_i(z) \, \mathrm{d}z = \frac{p_i}{y} \, \mathrm{d}z \,, \tag{4.3}$$

where the $p_i(z) = z^{N-1} + \cdots$ are monic polynomials of degree $N-1$ fixed by the conditions

$$\frac{1}{2i\pi} \oint_{\alpha_i} h_j = \delta_{ij} \,. \tag{4.4}$$

When the curve (3.5) is of genus $N-1$ (i.e. it is not degenerate), the h_is form a canonical basis of the vector space $\mathscr{L}^{(N)}$ defined by the following constraint on the divisor of one-forms on $\mathcal{C}$,

$$\mathscr{L}^{(N)} = \{\text{one-forms } \eta \mid (\eta) + \mu_0 + \hat{\mu}_0 \geq 0\} \,. \tag{4.5}$$

This corresponds to one-forms that are holomorphic except possibly at infinity on either sheet where they may have a simple pole. Note that the fact that the h_is are linearly independent follows from (4.4) and the fact that they generate $\mathscr{L}^{(N)}$ is a straightforward consequence of the Riemann-Roch theorem.

It is useful to understand the one-forms h_i also in the case of a degenerate curve of the form (3.20). Let us study what happens when two branch cuts join together, for example the branch cuts encircled by the contours α_1 and α_2. The genus of the curve then drops from $N-1$ to $N-2$. In the notation of (3.5) and (3.6), this corresponds to $x_1^+ = x_2^+$ or $x_1^- = x_2^-$ (we cannot have $x_1^\pm = x_2^\mp$ because P_+ and P_- do not have common roots). Let us choose for example $x_1^+ = x_2^+ = b_1$, and

$$y^2 = (z - b_1)^2 R_{2N-2}(z) \,. \tag{4.6}$$

Naively, the one-forms

$$h_i = \frac{p_i}{(z - b_1)\sqrt{R_{2N-2}}} \, \mathrm{d}z \tag{4.7}$$

then have poles at $z = b_1$ on the first and second sheets. However, this does not happen, because $p_j(b_1) = 0$ for all j. This follows from the constraints (4.4) for $i = 1$ or $i = 2$. Indeed, if we had $p_j(b_1) \neq 0$, then the contour integrals would have a logarithmic divergence in the degenerate limit. So we see that the h_i remains holomorphic at finite z on (4.7), with simple poles at infinity. In other words, the h_i belongs to the space $\mathscr{L}^{(N-1)}$ defined as in (4.5) but on the curve

$$y_{N-1}^2 = R_{2N-2}(z) \,. \tag{4.8}$$

Using (4.4), it follows that a canonical basis $\{h_i^{(N-1)}\}_{2 \leq i \leq N}$ of $\mathscr{L}^{(N-1)}$ is given by

$$h_1 = h_2 = h_2^{(N-1)} \,, \quad h_i = h_i^{(N-1)} \quad \text{for } i \geq 3 \,, \tag{4.9}$$

in the degenerate limit.

In the general case, (3.5) can degenerate to a genus $r-1$ curve

$$y_r^2 = R_{2r}(z) \tag{4.10}$$

with

$$y = H_{N-r}(z)\, y_r = \prod_{\ell=1}^{N-r} (z - b_\ell)\, y_r \,. \tag{4.11}$$

The h_is then generate the vector space $\mathscr{L}^{(r)}$ of one-forms on (4.10) that are holomorphic at finite z with at most simple poles at infinity, but with relations like $h_i = h_j$ depending on which cuts have joined. In particular, we have

$$p_i(b_\ell) = 0\,. \tag{4.12}$$

Clearly, the rank of the system $\{h_i\}$ is given by

$$\operatorname{rank}\{h_i\}_{1 \le i \le N} = r\,. \tag{4.13}$$

One-forms η_i. Let us introduce another basis of $\mathscr{L}^{(N)}$ given by

$$\eta_i = \phi_i(z)\,\mathrm{d}z = \frac{q_i}{y}\,\mathrm{d}z\,, \tag{4.14}$$

with

$$q_i(z) = \prod_{j \ne i}(z - x_j) = -\frac{\partial P(z)}{\partial x_i}\,. \tag{4.15}$$

We assume that the x_i are all distinct.

The one-forms η_i belong to $\mathscr{L}^{(r)}$ only for $r = N$ because of the $2(N - r)$ poles at $z = b_\ell$. Actually, even when the curve degenerates, the rank of the system $\{\eta_i\}$ doesn't change,

$$\operatorname{rank}\{\eta_i\}_{1 \le i \le N} = N\,. \tag{4.16}$$

This is a consequence of the linear independence of the polynomials q_i. Moreover, because the q_is form a basis for the polynomials of degree at most $N - 1$, there always exists a matrix A such that[3]

$$p_i(z) = \sum_{j=1}^{N} A_{ij}\, q_j(z)\,. \tag{4.17}$$

Using (4.3) and (4.14), we also have

$$\psi_i = \sum_{j=1}^{N} A_{ij}\, \phi_j\,, \quad h_i = \sum_{j=1}^{N} A_{ij}\, \eta_j\,. \tag{4.18}$$

The important point is that these relations are always valid, including in the cases where the curve degenerate, because the q_is are always linearly independent when the x_i are all distinct. Using (4.13) and (4.16), we also deduce that

$$\operatorname{rank} A = r\,. \tag{4.19}$$

Conversely, A of rank r clearly implies (4.13) which implies that the curve is of genus $r - 1$.

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[3]We show later that this definition is consistent with (4.1).

Variations of the Seiberg-Witten differential. Consider now the Seiberg-Witten differential

$$\lambda_{\mathrm{SW}} = zR(z)\,\mathrm{d}z\,. \tag{4.20}$$

An important property of $R(z)$ is that the solution to

$$\frac{F'(z)}{F(z)} = R(z) \tag{4.21}$$

is a function $F(z)$ defined on the Seiberg-Witten curve,

$$F(z; \mathbf{a}, q) = \langle \mathbf{a} | \det(z - X) | \mathbf{a} \rangle = \frac{1}{2}\Big(P(z) + \sqrt{P(z)^2 - 4q}\Big)\,. \tag{4.22}$$

Another useful identity is that

$$\frac{\delta F}{F} = \frac{\delta P}{y}\,, \tag{4.23}$$

where the variation δ is with respect to any parameter, for example the a_is or the x_is. To compute $\delta\lambda_{\mathrm{SW}}$, it is then convenient to write

$$\lambda_{\mathrm{SW}} = -\ln F\,\mathrm{d}z + \mathrm{d}(z\ln F)\,. \tag{4.24}$$

We get

$$\delta\lambda_{\mathrm{SW}} = -\frac{\delta F}{F}\,\mathrm{d}z + \mathrm{d}\Big(z\frac{\delta F}{F}\Big) = -\frac{\delta P}{y}\,\mathrm{d}z + \mathrm{d}\Big(z\frac{\delta P}{y}\Big)\,. \tag{4.25}$$

Note that when the curve is non-degenerate, $-\delta P \mathrm{d}z/y \in \mathscr{L}^{(N)}$.

We can use the above results to compute the derivatives of λ_{SW} with respect to a_i and to x_i. Using

$$\frac{1}{2i\pi} \oint_{\alpha_i} \frac{\partial\lambda_{\mathrm{SW}}}{\partial a_j} = \delta_{ij}\,, \tag{4.26}$$

which comes from taking the derivative of (3.9) with respect to a_j, we get, in terms of (4.3),

$$-\frac{1}{y}\frac{\partial P}{\partial a_i}\,\mathrm{d}z = h_i\,, \tag{4.27}$$

and thus

$$\frac{\partial\lambda_{\mathrm{SW}}}{\partial a_i} = h_i - \mathrm{d}(z\psi_i) = \psi_i\,\mathrm{d}z - \mathrm{d}(z\psi_i)\,. \tag{4.28}$$

Similarly, we get in terms of (4.14)

$$\frac{\partial\lambda_{\mathrm{SW}}}{\partial x_i} = \phi_i\,\mathrm{d}z - \mathrm{d}(z\phi_i)\,. \tag{4.29}$$

The tangent space to Σ_r. Let us define Σ_r to be the r-dimensional surface in $\mathbf{x}$-space on which the Seiberg-Witten curve degenerates to a genus $r-1$ surface. Let us show that the vectors

$$e_i = \sum_{j=1}^{N} A_{ij}\frac{\partial}{\partial x_j} \tag{4.30}$$

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generate the tangent space to Σ_r. Due to (4.19), all we have to show is that $e_i(\mathbf{x}) \in T_{\mathbf{x}}\Sigma_r$. The result is true essentially by construction, but let us see explicitly how it works.

The surface Σ_r is defined by the equation (3.20). Thus an arbitrary vector $\nabla \in T_{\mathbf{x}}\Sigma_r$ if and only if there exists polynomials H_{N-r} and R_{2r} such that

$$\nabla \cdot \left(P(z)^2 - 4q\right) = \nabla \cdot \left(H_{N-r}(z)^2 R_{2r}(z)\right) \tag{4.31}$$

or equivalently

$$2P\nabla \cdot P = H_{N-r}\left(2R_{2r}\nabla \cdot H_{N-r} + H_{N-r}\nabla \cdot R_{2r}\right). \tag{4.32}$$

Equation (3.20) implies that P and H_{N-r} cannot have common roots. Thus (4.32) implies that

$$\nabla \cdot P = H_{N-r}Q_{r-1}, \tag{4.33}$$

$$2PQ_{r-1} = 2R_{2r}\nabla \cdot H_{N-r} + H_{N-r}\nabla \cdot R_{2r}, \tag{4.34}$$

for some degree $r-1$ polynomial Q_{r-1}. Conversely, assume that the vector ∇ is such that (4.33) is satisfied. Then the equation (4.34) can be viewed as a constraint that determines the polynomials $\nabla \cdot H_{N-r}$ and $\nabla \cdot R_{2r}$ (the equation is of degree $N+r-1$, for $N+r$ unknown in $\nabla \cdot H_{N-r}$ and $\nabla \cdot R_{2r}$).

So all we have to show is that the vectors (4.30) satisfy

$$e_i \cdot P = H_{N-r}p_i^{(r)} = \prod_{\ell=1}^{N-r}(z - b_\ell)p_i^{(r)} \tag{4.35}$$

for some polynomials $p_i^{(r)}$ of degrees $r-1$. This follows immediately from (4.15) and (4.17), which imply that

$$e_i \cdot P = -p_i(z), \tag{4.36}$$

and from (4.12). Finally, we have derived that

$$T_{\mathbf{x}}\Sigma_r = \text{Vect}\left[e_i\right]_{1 \leq i \leq N}. \tag{4.37}$$

4.2 Solving $dW_{\mathbf{mic}} = 0$

Let us rewrite (3.14) in the form

$$W_{\text{mic}} = \frac{1}{2i\pi} \oint_\alpha \frac{W(z)}{z} \lambda_{\text{SW}}. \tag{4.38}$$

Using (4.28) and performing an integration by part, we get

$$\frac{\partial W_{\text{mic}}}{\partial a_i} = \frac{1}{2i\pi} \oint_\alpha W'(z)\psi_i(z)\,dz. \tag{4.39}$$

Similarly, from (3.22) and (4.29) we obtain

$$\frac{\partial W_{\text{SC}}^{(N)}}{\partial x_i} = W'(x_i) = \frac{1}{2i\pi} \oint_\alpha W'(z)\phi_i(z)\,dz. \tag{4.40}$$

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The case of distinct x_i. Let us assume for the moment that the x_i are all distinct. Then using (4.18), (4.39) and (4.40), we find (4.1). This equation is valid for any genus $r-1$ of the Seiberg-Witten curve, and can be written in terms of the vector fields (4.30) as

$$e_i \cdot W_{\mathrm{SC}}^{(N)} = 0 \,. \tag{4.41}$$

The most general solution is labeled by r, and, for a given r, we find using (4.37) that it corresponds to extrema of $W_{\mathrm{SC}}^{(N)}$ on the surface Σ_r. This is exactly the prescription used in the strong coupling approach. Using (4.39) and (4.12), finding the extrema along Σ_r is equivalent to imposing

$$\oint_\alpha \frac{W'(z)}{y_r} Q_{r-1} \, \mathrm{d}z = 0 \tag{4.42}$$

for any degree $r-1$ polynomial Q_{r-1}. Since the integral in (4.42) simply picks the simple pole at infinity, this yields the condition

$$\frac{W'}{y_r} = \tilde{H}_{d-r} + \mathcal{O}(1/z^{r+1}) \tag{4.43}$$

for some degree $d-r$ polynomial $\tilde{H}_{d-r}$, or

$$W' = \tilde{H}_{d-r} y_r + \mathcal{O}(1/z) \,. \tag{4.44}$$

Taking the square of (4.44), we find

$$W'^2 = \tilde{H}_{d-r}^2 R_{2r} + \mathcal{O}(z^{d-1}) \,. \tag{4.45}$$

Since both W'^2 and $\tilde{H}_{d-r}^2 R_{2r}$ are polynomials, this is equivalent to the existence of a degree $d-1$ polynomial Δ_{d-1} such that

$$W'(z)^2 - \Delta_{d-1}(z) = H_{d-r}(z)^2 R_{2r}(z) \,. \tag{4.46}$$

This is the usual factorization condition which, together with (3.20), yields the full solution of the theory.

The case of $x_i = x_j$. It might happen that, for some particular values of the couplings, some solutions correspond to $x_i = x_j$ for a pair of distinct indices i and j. The previous analysis doesn't apply immediately in this case because for $x_i = x_j$, we have $q_i = q_j$, thus the polynomials q_k are no longer independent and the matrix A is not well-defined. However, let us show that the formulas behave smoothly when we approach such a point. Let us start from a case where x_i and x_j are very close to each other,

$$x_i - x_j = \epsilon \,. \tag{4.47}$$

Then $q_i - q_j \sim \epsilon$. Using (4.17), we see that in the limit $\epsilon \to 0$, the components A_{ki} and A_{kj} diverge as

$$A_{ki} \sim \frac{b_{ki}}{\epsilon} \sim -A_{kj} \,. \tag{4.48}$$

On the other hand, the potentially diverging terms in (4.1) read

$$A_{ki} \frac{\partial W_{\mathrm{SC}}^{(N)}}{\partial x_i} + A_{kj} \frac{\partial W_{\mathrm{SC}}^{(N)}}{\partial x_j} \sim \frac{b_{ki}}{\epsilon} \big(W'(x_i) - W'(x_j) \big) \sim b_{ki} W''(x_i) \,, \tag{4.49}$$

and thus the limit $\epsilon \to 0$ is smooth.

5. Conclusion and outlook

We have shown that a microscopic approach to $\mathcal{N} = 1$ gauge theories, based on Nekrasov's instanton technology, is possible. The stationary points of the microscopic superpotential $W_{\mathrm{mic}}(\mathbf{a})$ yield all the quantum vacua, including the strongly coupled confining vacua, and the result is consistent with the strong coupling approach or the Dijkgraaf-Vafa matrix model. In particular, at the extrema of W_{mic}, the gauge theory resolvent (3.7), (3.8) precisely coincide with the prediction of the matrix model. In other word, we have obtained a full microscopic description of the expectation values $\langle \mathrm{Tr}\, X^k \rangle$ in all the vacua of the theory.

One of the most interesting potential application of the superpotential W_{mic} is the non-perturbative study of the generalized Konishi anomaly equations. At the moment, only a perturbative analysis of these equations has appeared [10], whereas the equations are supposed to be valid at the non-perturbative level (see for example [21] for a discussion). On general grounds, one may expect to have relations like

$$\delta W_{\mathrm{mic}} = \mathscr{A} \,, \qquad (5.1)$$

where $\mathscr{A}$ is the anomaly polynomial and δ is a suitable variation. At the perturbative level, the variations δ one must consider [10] act on the fields as $\delta X \sim X^{n+1}$, $\delta X \sim W^\alpha W_\alpha X^{n+1}$, and thus generate a sort of super Virasoro algebra. At the non-perturbative level, the variations δ and associated algebra must be quantum corrected (this happens because the transformations are non-linear). The corrections can in principle be studied starting from (5.1).

To complete the above program, we need to study in full details the glueball operators

$$v_k(\mathbf{a}, \mathbf{g}, q) = -\frac{1}{16\pi^2} \big\langle \mathbf{a} \big| \mathrm{Tr}\, W^\alpha W_\alpha X^k \big| \mathbf{a} \big\rangle \qquad (5.2)$$

or the associated generating function

$$S(z; \mathbf{a}, \mathbf{g}, q) = \sum_{k \geq 0} \frac{v_k(\mathbf{a}, \mathbf{g}, q)}{z^{k+1}} \,. \qquad (5.3)$$

The function $S(z; \mathbf{a}, \mathbf{g}, q)$ is not known for arbitrary values of the boundary conditions $\mathbf{a}$. As sketched in 3.1, it depends on subleading corrections in ϵ in Nekrasov's formalism. Computing this function, showing that it enters the anomaly equations (5.1) at the non-perturbative level as derived in [10] in perturbation theory, and that it coincides with the matrix model prediction at the extrema of W_{mic} will be a central topic in forthcoming publications [11, 12].

Acknowledgments

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